1956-01

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— BY —

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A REPORT
TO THE NAVY DEPARTMENT
BUREAU OF SHIPS
UPON AN INVESTIGATION CONDUCTED UNDER
BUSHIPS PROJECT ORDER NO. 10799/55

JANUARY, 1956

TECHNICAL REPORT NO. 15

UNITED STATES

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INTRODUCTION

Broadly speaking, it may be stated that the calculus of variations in the theory of functions of real variables serves as a tool for discovering certain physical laws. The integrand of a definite integral is perturbed by adding to it a parameter times a function of the independent variable, subject to the condition that the function vanishes at both limits of the integration. The physical laws are obtained by requiring the first derivative of the perturbed integral with respect to the parameter to vanish. Perhaps the most useful laws are those for which the integral is made a minimum by the value of the parameter thus obtained.

Recently, the calculus of variations has been applied to many physical problems set up in terms of analytic functions of a complex variable. In particular, Storer[1] and Tai[2] have used it in obtaining a first order solution to the symmetrically driven straight cylindrical antenna.

In agreement with Storer[1], if

$$W = f(z) = U(x,y) + j\, V(x,y)$$

is analytic, and if W_0 is a constant, using the Cauchy-Riemann equations, it is not difficult to show that setting

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1. J. E. Storer, "Variational Solution to the Problem of the Symmetrical Antenna", Cruft Lab., Harvard University, Cambridge, Mass., Tech. Rept. No. 101, 1950

2. C. T. Tai, "A Variational Solution to the Problem of Cylindrical Antennas", Stanford Res. Inst., Stanford, Calif., Tech.Rept. No.12 1950

$$\frac{\partial}{\partial x} |W-W_o| = 0, \quad \frac{\partial}{\partial y} |W-W_o| = 0$$

is equivalent to setting

$$\frac{dW}{dz} = 0,$$

or that the first term of a Taylor's expansion of the function vanishes.

However, by taking the second derivatives and using the Cauch-Riemana equations in connection with the standard tests for maxima and minima of functions of two variables[3], it can be shown that $|W-W_o|$ is neither a maximum nor a minimum when the first derivative with respect to z vanishes. That is, if Z_o is the true input impedance of an antenna, and if $Z(\epsilon_r + j\,\epsilon_i) = Z(\epsilon)$ is an impedance obtained by using $I(\epsilon, x) = I(x) + \eta\,\epsilon(x)$ as an approximation to the true current distribution function $I(x)$, the requirement

$$\frac{dZ(\epsilon)}{d\epsilon} = 0$$

does not minimize $|Z-Z_o|$, as implied by Storer[1], but merely yields a minimax[4] for the value $\epsilon = \epsilon_o$ found by solving the above equation.

3. E. B. Wilson, "Advanced Calculus", Ginn and Co., New York, N.Y., pp 114-115, 1912.

4. Wilson, ibid., p.115.

PHYSICAL SIGNIFICANCE OF VARIATIONAL METHOD

The variational method postulates the generalized Kirchoff's law[5].

$$\frac{j30}{k}\int_{-\ell}^{\ell} I(x')G(x',x)dx' = Z_o I(0)\,\delta(x) \qquad (1)$$

in which

ℓ = the half length of the antenna

a = the radius of the antenna

$k = \frac{2\pi}{\lambda}$

$$G(x',x) = \left(\frac{d^2}{dx^2} + k^2\right)\frac{e^{-jkr(x',x)}}{r(x',x)}$$

and $\delta(x)$ is the Dirac delta impulse function.

The driving point impedance is then

chosen in the form

$$Z_o = \frac{j30}{kI^2(0)}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} I(x)I(x')G(x',x)dx'dx \qquad (2)$$

It has been shown[6] that this form, using the approximate

5. Tai, op.cit., p.4

6. C. T. Tai, "A new interpretation of the integral equation formulation of cylindrical antennas", IRE Trans., PGAP, Vol. AP3; pp. 125-127, July, 1955

Green's function with $r = \left[(x-x')^2 + a^2\right]^{\frac{1}{2}}$ yields the same result as that obtained by using the exact Green's function.

The current function is perturbed by choosing

$$I(x,A) = I(x) + \epsilon\,\eta(x) = I_o\left[\,f(x) + A\,\eta(x)\right] \tag{3}$$

If A_o is a root of $\dfrac{d\,Z_o(A)}{dA} = 0$, the impedance is found to be[7]

$$Z_o(A_o) = 30\;\frac{V_{11}\,V_{22} - V_{12}^2}{f^2(0)V_{22} - 2f(0)\,\eta(0)V_{12}\,\eta^2(0)V_{11}} \tag{4}$$

with

$$\begin{aligned}
V_{11} &= \frac{j}{k}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} f(x)f(x')G(x',x)dx'dx \\
V_{12} &= \frac{j}{k}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} f(x)\,\eta(x')G(x',x)dx'dx \\
V_{22} &= \frac{j}{k}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \eta(x)\,\eta(x')G(x',x)dx'dx
\end{aligned} \tag{5}$$

By simple algebraic manipulation, equation (4) may be converted into

$$Z_{in} = \frac{Z_{11}Z_{22} - Z_{12}^2}{Z_{11} + Z_{22} - 2Z_{12}} \tag{6}$$

with

$$\begin{aligned}
Z_{11} &= \frac{j30}{kI^2(0)}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} I_1(x)I_1(x')G(x',x)dx'dx \\
Z_{12} &= \frac{j30}{kI_1(0)I_2(0)}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} I_1(x)I_2(x')G(x',x)dx'dx \\
Z_{22} &= \frac{j30}{kI_2^2(0)}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} I_2(x)I_2(x')G(x',x)dx'dx
\end{aligned} \tag{7}$$

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7. Tai, Rept., No. 12, op. cit., p 11.

in which it is assumed that neither $I_1(0)$ nor $I_2(0)$ is zero.

Equation (6) is recognized as the solution of the mesh equations,

$$Z_{11}I_1(0) + Z_{12}I_2(0) = V_o$$
$$Z_{21}I_1(0) + Z_{22}I_2(0) = V_o \tag{8}$$

The current distribution becomes

$$I(x) = \frac{V_o}{I_1(0)} \frac{Z_{22}-Z_{12}}{Z_{11}Z_{22}-Z_{12}^2} I_1(x) + \frac{V_o}{I_2(0)} \frac{Z_{11}-Z_{12}}{Z_{11}Z_{22}-Z_{12}^2} I_2(x) \tag{9}$$

Suppose that $I_2(0) \neq 0$ with $I_1(0) = 0$. Let

$$\zeta_{11} = \lim_{x\to 0} \frac{I_1^2(x)}{I_{01}^2} Z_{11}, \quad \zeta_{12} = \lim_{x\to 0} \frac{I_1(x)Z_{12}}{I_{01}}$$

Then equations (6) and (9) become, respectively

$$Z_{in} = Z_{22} - \frac{\zeta_{12}^2}{\zeta_{11}}, \tag{10}$$

and

$$I(x) = \frac{V_o}{\zeta_{11}Z_{22} - \zeta_{12}^2} \left[\frac{\zeta_{11}}{I_2(0)} I_2(x) - \frac{\zeta_{12}I_1(x)}{I_{01}} \right] \tag{11}$$

in which I_{01}^{-1} is a normalizing factor.

From equations (6), (8), and (9), it becomes evident that the vanishing of the first term in the Taylor's expansion of the perturbed current simply yields the physical information that two currents have been postulated to exist in parallel along the antenna. From equations (10) and (11), it becomes further evident that for

singular cases one current becomes the <u>feed current</u> with the other parasitically excited.

This recognition greatly simplifies the variational theory of cylindrical antennas. In fact, the statement might be ventured that the variational method is not a distinct method *per se*. Any scheme for finding the self and mutual impedances of the two postulated current distributions may be used. Since $|Z_{in}-Z_o|$ is not minimized, there is no *a priori* reason for assuming that the driving point impedance computed by a method with a vanishing first term in its Taylor's expansion is any more accurate than that computed from the parallel circuit with the self and mutual impedances being computed by any recognized method.

FIRST ORDER SOLUTION BY THE GENERALIZED CIRCUIT

For two straight currents, the mutual impedance by the generalized circuit[8] scheme is given by

$$Z_{12} = \frac{j30}{k\,|f(0)|^2}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \mathrm{Re}\left[f_1(x)f_2(x')^*\right]\left[\frac{\partial^2}{\partial x^2}+k^2\right]\frac{e^{-jkr_{12}}}{r_{12}}\,dx\,dx', \qquad (12)$$

in which the operator Re takes the real part of the product of one current by the complex conjugate of the other current. The current functions are normalized at the driving point. In case the current functions are different and at least one of them is not real, the

8. J. G. Chaney, "A critical study of the circuit concept", Jour-Appl.Phys., Vol.22; 1429-1436, Dec., 1951.

reciprocity theorem requires the use of only the real part of the current product.

It is possible to show that the results obtained with the approximate Kernel are the same as those obtained when the exact Kernel is used in equation (12).

For a relatively low loss transmission line, Tai[9] approximated the current distribution function by

$$I(x) = I_o \left[\mathrm{Sin}\, \beta\,(\ell - x) - j\,\alpha\,(\ell - x)\cos\beta\,(\ell - x) \right. \qquad (13)$$

from which he chose his pair of currents. The current on a lossy line also may be written

$$I(x) = I_o \left[e^{\alpha x}\sin\beta\,(\ell - x) - j\, e^{\alpha\ell}\sinh\alpha\,(\ell - x)e^{j\beta(\ell - x)} \right.$$

$$\approx I_o \left[\sin\beta(\ell - x) - j\alpha e^{\gamma\ell}\,(\ell - x)e^{-j\beta x} \right. \quad . \qquad (14)$$

To the writer, this arrangement intuitively seems preferable to that of equation (13). Hence, the current pair is chosen as

$$\begin{aligned} I_1(x) &= I_{01}\sin k(\ell - |x|) \\ I_2(x) &= I_{02}\,\alpha\,(\ell - |x|)e^{-jk|x|} \end{aligned} \qquad (15)$$

The self impedance Z_{11} is then the well known zero order solution by the classical induced emf method[6]. For the others,

$$Z_{22} = \frac{j30}{k}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\left(1 - \left|\frac{x}{\ell}\right|\right)\left(1 - \left|\frac{x'}{\ell}\right|\right)e^{jk(|x| - |x'|)}\left(\frac{\partial^2}{\partial x^2} + k^2\right)\frac{e^{-jkr_{12}}}{r_{12}}\,dx\,dx' \qquad (16)$$

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9. Tai, Rept. No. 12, op. cit., p. 12

$$Z_{12} = \frac{j30}{k\sin k\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} (1-|\frac{x}{\ell}|)\sin k(\ell-|x'|)\cos kx \left(\frac{\partial^2}{\partial x^2}+k^2\right)\frac{e^{-jkr_{12}}}{r_{12}}dxdx' \qquad (17)$$

It is not necessary to take the real part in equation (16) because it is a self impedance and the two current functions are identical. The resulting formulas are listed in Appendix A for $|ka|\ll 1$, $2a \ll \ell$.

The current distribution along the antenna then becomes,

$$I(x) = \frac{V_o}{Z_{11}Z_{22}-Z_{12}^2}\left[\frac{Z_{22}-Z_{12}}{\sin k\ell} \sin k(\ell-|x|)+(Z_{11}-Z_{12})(1-|\frac{x}{\ell}|)e^{-jk|x|}\right] \qquad (18)$$

ASYMTOTIC VALUES

It is of interest to examine the asymtotic values of the current and of the impedance.

For $\Omega = 2\ \ln\frac{2\ell}{a} \to \infty$

$$Z_{11} = Z_{12} \to -j60\,\Omega\cot k\ell\ ,\ Z_{22} \to -j\frac{60}{k\ell}\Omega\ ,\ Z_{in} \to Z_{11},$$

and

$$I(x) \to \frac{V_o}{Z_{11}\sin k\ell}\ \sin k(\ell-|x|) = j\frac{V_o}{60\Omega}\ \frac{\sin k(\ell-|x|)}{\cos k\ell}$$

For $k\ell \ll 1$,

$$Z_{11} = Z_{12} = Z_{22} \to 20(k\ell)^2 - j\frac{60}{k\ell}\ (\Omega-2-2\ \ln 2)$$

and

$$I(x) \to \frac{V_o}{Z_{11}}\ (1-|\frac{x}{\ell}|)$$

with $Z_{in} \to Z_{11}$.

For $k\ell \to n\pi$, $n = 1, 2, 3, - - -$,

$$Z_{in} = Z_{22} - \frac{\zeta_{12}^2}{\zeta_{11}}$$

and

$$I(x) = \frac{V_o}{\zeta_{11} Z_{22} - \zeta_{12}^2} \left[\zeta_{11} (1 - |\tfrac{x}{\ell}|) e^{-jk|x|} - \zeta_{12} \sin k(\ell - |x|) \right. \qquad (19)$$

From equation (19), it is seen that the attenuated travelling wave becomes the <u>feed current</u>, and that the standing wave becomes parasitically excited.

Thus asymtotically, this method is equivalent to the induced emf method with the removal of the singularities.

CONCLUSION

Some values of the driving point impedance computed by this method (Table 1) were plotted on curves by Tai[10] for comparison with values computed by the King-Middleton, the variational, and the Schelkunoff methods. They show excellent correlation with the other methods, being very close to the results of Tai with perhaps about as many of the excursions being on the Schelkunoff side of the curves, as on the King-Middleton side of the curves.

Table I

	$\Omega = 10$	$\Omega = 15$
$k\ell$	Z_{in}	Z_{in}
$\pi/2$	83.0 + j41.8	77.5 + j42.3
2.2	493 + 304	360 + j654
2.6	883 - j94	1282 + j1101
2.9	633 - j492	2513 - j349
π	372 - j493	1345 - j1447
$3\pi/2$	78.3 + j15.0	92 + j2
5.1	178 + j104	288 + j287
5.6	586 + j63	871 + j683
6.0	340 - j262	1699 - j130
2π	332 - j263	1094 - j904

10. Tai, ibid, Figs. 4-5.

Table 2

	$k\ell = \pi/2$	$k\ell = \pi$
	Z_{in}	Z_{in}
10	83.0 + j41.8	372 - j493
12	79.8 + j42.0	683 - 837
15	77.5 + j42.3	1345 - 1447
22	75.6 + j42.3	3797 - j3463

The comparative simplicity of the method of obtaining the first order generalized circuit solution of the cylindrical antenna should justify its introduction into an already crowded group of antenna theories. It should be pointed out that it is not a variational method but that the variational method is a special case of the generalized circuit method. It is believed that the simplicity of this method will greatly facilitate the teaching of the problem of finding the driving point impedance of the cylindrical antenna.

APPENDIX A

$$Z_{11}\sin^2 k\ell = 60\,\mathrm{cin}\,2k\ell - 30\sin 2k\ell(2\,\mathrm{Si}\,2k\ell - \mathrm{Si}\,4k\ell)$$
$$+30\cos 2k\ell(2\,\mathrm{cin}\,2k\ell - \mathrm{cin}\,4k\ell)$$
$$+j\{60\,\mathrm{Si}\,2k\ell - 30\sin 2k\ell(\Omega - \ln 4 + \mathrm{cin}\,4k\ell - 2\,\mathrm{cin}\,2k\ell)$$
$$+30\cos 2k\ell(2\,\mathrm{Si}\,2k\ell - \mathrm{Si}\,4k\ell)\}$$

$$Z_{12}\sin k\ell = 60\cos k\ell(\mathrm{Si}\,4k\ell - 2\,\mathrm{Si}\,2k\ell) + 60\sin k\ell(\mathrm{cin}\,4k\ell - \mathrm{cin}\,2k\ell)$$

$$+j\{60\cos k\ell(\ln 4 - \Omega + 1 - \mathrm{cin}\,4k\ell + 2\,\mathrm{cin}\,2k\ell)$$
$$+60\sin k\ell(\mathrm{Si}\,4k\ell - \mathrm{Si}\,2k\ell) + \frac{15}{k\ell}(3\sin k\ell + 2\sin 2k\ell - \sin 3k\ell)\}$$

$$Z_{22} = 60\,\mathrm{cin}\,2k\ell - 30 + \frac{30}{k\ell}(\sin 2k\ell - 2\,\mathrm{Si}\,2k\ell)$$
$$+\frac{15}{(k\ell)^2}\Big[1 + 2\,\mathrm{cin}\,2k\ell + \cos 2k\ell(2\,\mathrm{cin}\,2k\ell - 2\,\mathrm{cin}\,4k\ell - 1)$$
$$+2\sin 2k\ell(\mathrm{Si}\,4k\ell - \mathrm{Si}\,2k\ell)\Big]$$

$$+j\{60\,\mathrm{Si}\,2k\ell - \frac{30}{k\ell}(2\Omega - 2 - \cos 2k\ell - 2\,\mathrm{cin}\,2k\ell)$$
$$+\frac{15}{(k\ell)^2}\Big[2\,\mathrm{Si}\,2k\ell + \sin 2k\ell(2\ln 4 + 1 + 2\,\mathrm{cin}\,2k\ell - 2\,\mathrm{cin}\,4k\ell)$$
$$+2\cos 2k\ell(\mathrm{Si}\,2k\ell - \mathrm{Si}\,4k\ell)\Big]\}$$

$$\mathrm{Si}\,x = \int_0^x \frac{\sin t}{t}\,dt, \quad \mathrm{Ci}\,x = \int_\infty^x \frac{\cos t}{t}\,dt,$$
$$\mathrm{cin}\,x = \int_0^x \frac{1-\cos t}{t}\,dt = \ln \gamma x - \mathrm{Ci}\,x, \quad \ln\gamma = 0.5772\cdots$$